FM3003 - Calculus III

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Course website

http://fm3003.wikispaces.com

1 Vectors in Euclidean Space

1.1 Introduction

In single-variable calculus, the functions that one encounters are functions of a variable (usually x or t) that varies over some subset of the real number line (which we denote by $\mathbb{R}$). For such a function, say, $y = f(x)$, the **graph** of the function f consists of the points $(x, y) = (x, f(x))$. These points lie in the **Euclidean plane**, which, in the **Cartesian** or **rectangular** coordinate system, consists of all ordered pairs of real numbers (a, b). We use the word "Euclidean" to denote a system in which all the usual rules of Euclidean geometry hold. We denote the Euclidean plane by $\mathbb{R}^2$; the "2" represents the number of *dimensions* of the plane. The Euclidean plane has two perpendicular **coordinate axes**: the x-axis and the y-axis.

In vector (or multivariable) calculus, we will deal with functions of two or three variables (usually x, y or x, y, z, respectively). The graph of a function of two variables, say, $z = f(x, y)$, lies in **Euclidean space**, which in the Cartesian coordinate system consists of all ordered triples of real numbers (a, b, c). Since Euclidean space is 3-dimensional, we denote it by $\mathbb{R}^3$. The graph of f consists of the points $(x, y, z) = (x, y, f(x, y))$. The 3-dimensional coordinate system of Euclidean space can be represented on a flat surface, such as this page or a blackboard, only by giving the illusion of three dimensions, in the manner shown in Figure 1.1.1. Euclidean space has three mutually perpendicular coordinate axes (x, y and z), and three mutually perpendicular coordinate planes: the xy-plane, yz-plane and xz-plane (see Figure 1.1.2).

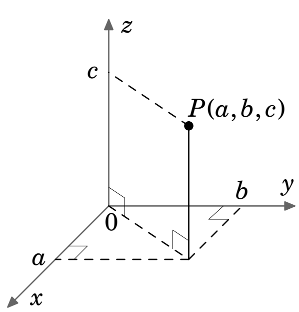


Figure 1.1.1

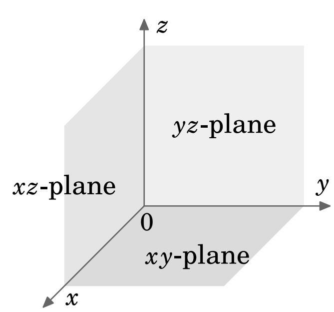


Figure 1.1.2

The coordinate system shown in Figure 1.1.1 is known as a **right-handed coordinate system**, because it is possible, using the right hand, to point the index finger in the positive direction of the x-axis, the middle finger in the positive direction of the y-axis, and the thumb in the positive direction of the z-axis, as in Figure 1.1.3.

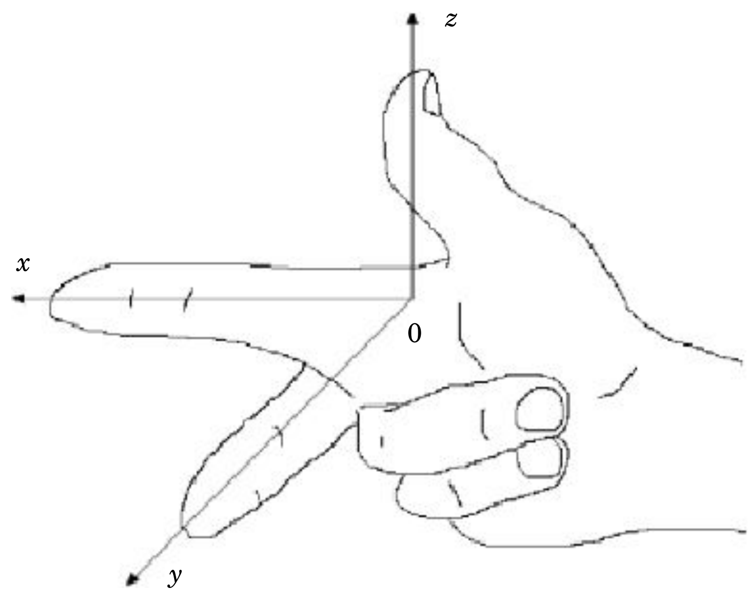


Figure 1.1.3 Right-handed coordinate system

An equivalent way of defining a right-handed system is if you can point your thumb upwards in the positive z-axis direction while using the remaining four fingers to rotate the x-axis towards the y-axis. Doing the same thing with the left hand is what defines a **left-handed coordinate system**. Notice that switching the x- and y-axes in a right-handed system results in a left-handed system, and that rotating either type of system does not change its "handedness". Throughout the book we will use a right-handed system.

For functions of three variables, the graphs exist in 4-dimensional space (i.e. $\mathbb{R}^4$), which we can not see in our 3-dimensional space, let alone simulate in 2-dimensional space. So we can only think of 4-dimensional space abstractly. For an entertaining discussion of this subject, see the book by ABBOTT.[1]

So far, we have discussed the *position* of an object in 2-dimensional or 3-dimensional space. But what about something such as the velocity of the object, or its acceleration? Or the gravitational force acting on the object? These phenomena all seem to involve motion and *direction* in some way. This is where the idea of a *vector* comes in.

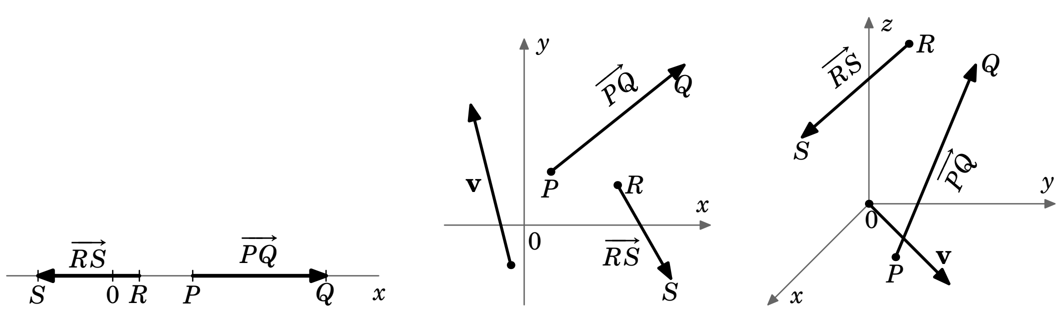
[1] One thing you will learn is why a 4-dimensional creature would be able to reach inside an egg and remove the yolk without cracking the shell!

You have already dealt with velocity and acceleration in single-variable calculus. For example, for motion along a straight line, if $y = f(t)$ gives the displacement of an object after time t, then $dy/dt = f'(t)$ is the velocity of the object at time t. The derivative $f'(t)$ is just a number, which is positive if the object is moving in an agreed-upon "positive" direction, and negative if it moves in the opposite of that direction. So you can think of that number, which was called the velocity of the object, as having two components: a *magnitude*, indicated by a nonnegative number, preceded by a *direction*, indicated by a plus or minus symbol (representing motion in the positive direction or the negative direction, respectively), i.e. $f'(t) = \pm a$ for some number $a \ge 0$. Then a is the magnitude of the velocity (normally called the *speed* of the object), and the $\pm$ represents the direction of the velocity (though the $+$ is usually omitted for the positive direction).

For motion along a straight line, i.e. in a 1-dimensional space, the velocities are also contained in that 1-dimensional space, since they are just numbers. For general motion along a curve in 2- or 3-dimensional space, however, velocity will need to be represented by a multi-dimensional object which should have both a magnitude and a direction. A geometric object which has those features is an arrow, which in elementary geometry is called a "directed line segment". This is the motivation for how we will define a vector.

Definition 1.1. A (nonzero) **vector** is a directed line segment drawn from a point P (called its **initial point**) to a point Q (called its **terminal point**), with P and Q being distinct points. The vector is denoted by $\overrightarrow{PQ}$. Its **magnitude** is the length of the line segment, denoted by $\|\overrightarrow{PQ}\|$, and its **direction** is the same as that of the directed line segment. The **zero vector** is just a point, and it is denoted by $\mathbf{0}$.

To indicate the direction of a vector, we draw an arrow from its initial point to its terminal point. We will often denote a vector by a single bold-faced letter (e.g. $\mathbf{v}$) and use the terms "magnitude" and "length" interchangeably. Note that our definition could apply to systems with any number of dimensions (see Figure 1.1.4 (a)-(c)).



(a) One dimension **(b)** Two dimensions **(c)** Three dimensions

Figure 1.1.4 Vectors in different dimensions

A few things need to be noted about the zero vector. Our motivation for what a vector is included the notions of magnitude and direction. What is the magnitude of the zero vector? We define it to be zero, i.e. $\|\mathbf{0}\| = 0$. This agrees with the definition of the zero vector as just a point, which has zero length. What about the direction of the zero vector? A single point really has no well-defined direction. Notice that we were careful to only define the direction of a *nonzero* vector, which is well-defined since the initial and terminal points are distinct. Not everyone agrees on the direction of the zero vector. Some contend that the zero vector has *arbitrary* direction (i.e. can take any direction), some say that it has *indeterminate* direction (i.e. the direction can not be determined), while others say that it has *no* direction. Our definition of the zero vector, however, does not require it to have a direction, and we will leave it at that.[2]

Now that we know what a vector is, we need a way of determining when two vectors are equal. This leads us to the following definition.

Definition 1.2. Two nonzero vectors are **equal** if they have the same magnitude and the same direction. Any vector with zero magnitude is equal to the zero vector.

By this definition, vectors with the same magnitude and direction but with different initial points would be equal. For example, in Figure 1.1.5 the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ all have the same magnitude $\sqrt{5}$ (by the Pythagorean Theorem). And we see that $\mathbf{u}$ and $\mathbf{w}$ are parallel, since they lie on lines having the same slope $\frac{1}{2}$, and they point in the same direction. So $\mathbf{u} = \mathbf{w}$, even though they have different initial points. We also see that $\mathbf{v}$ is parallel to $\mathbf{u}$ but points in the opposite direction. So $\mathbf{u} \neq \mathbf{v}$.

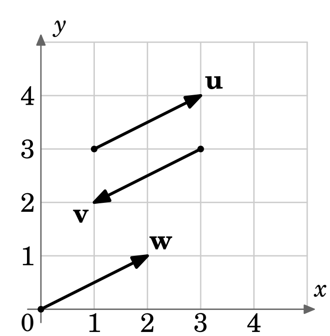


Figure 1.1.5

So we can see that there are an infinite number of vectors for a given magnitude and direction, those vectors all being equal and differing only by their initial and terminal points. Is there a single vector which we can choose to represent all those equal vectors? The answer is yes, and is suggested by the vector $\mathbf{w}$ in Figure 1.1.5.

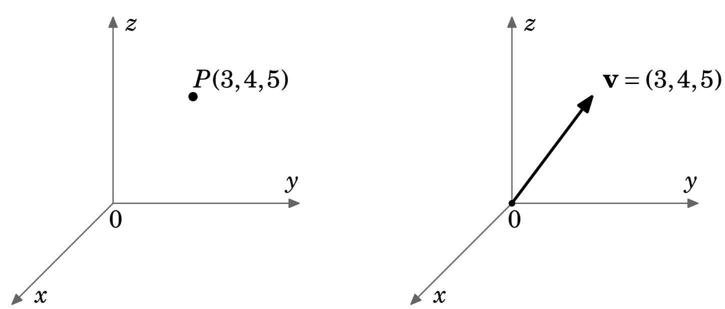
[2]In the subject of linear algebra there is a more abstract way of defining a vector where the concept of "direction" is not really used. See ANTON and RORRES.

Unless otherwise indicated, when speaking of "the vector" with a given magnitude and direction, we will mean the one whose initial point is at the origin of the coordinate system.

Thinking of vectors as starting from the origin provides a way of dealing with vectors in a standard way, since every coordinate system has an origin. But there will be times when it is convenient to consider a different initial point for a vector (for example, when adding vectors, which we will do in the next section).

Another advantage of using the origin as the initial point is that it provides an easy correspondence between a vector and its terminal point.

Example 1.1. Let $\mathbf{v}$ be the vector in $\mathbb{R}^3$ whose initial point is at the origin and whose terminal point is $(3,4,5)$. Though the *point* $(3,4,5)$ and the vector $\mathbf{v}$ are different objects, it is convenient to write $\mathbf{v} = (3,4,5)$. When doing this, it is understood that the initial point of $\mathbf{v}$ is at the origin $(0,0,0)$ and the terminal point is $(3,4,5)$.



(a) The point (3,4,5) **(b)** The vector (3,4,5)

Figure 1.1.6 Correspondence between points and vectors

Unless otherwise stated, when we refer to vectors as $\mathbf{v} = (a,b)$ in $\mathbb{R}^2$ or $\mathbf{v} = (a,b,c)$ in $\mathbb{R}^3$, we mean vectors in Cartesian coordinates starting at the origin. Also, we will write the zero vector $\mathbf{0}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ as $(0,0)$ and $(0,0,0)$, respectively.

The point-vector correspondence provides an easy way to check if two vectors are equal, without having to determine their magnitude and direction. Similar to seeing if two points are the same, you are now seeing if the terminal points of vectors starting at the origin are the same. For each vector, find the (unique!) vector it equals whose initial point is the origin. Then compare the coordinates of the terminal points of these "new" vectors: if those coordinates are the same, then the original vectors are equal. To get the "new" vectors starting at the origin, you *translate* each vector to start at the origin by subtracting the coordinates of the original initial point from the original terminal point. The resulting point will be the terminal point of the "new" vector whose initial point is the origin. Do this for each original vector then compare.

Example 1.2. Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in $\mathbb{R}^3$, where $P=(2,1,5), Q=(3,5,7), R=(1,-3,-2)$ and $S=(2,1,0)$. Does $\overrightarrow{PQ}=\overrightarrow{RS}$?
Solution: The vector $\overrightarrow{PQ}$ is equal to the vector $\mathbf{v}$ with initial point $(0,0,0)$ and terminal point $Q-P=(3,5,7)-(2,1,5)=(3-2,5-1,7-5)=(1,4,2)$.
Similarly, $\overrightarrow{RS}$ is equal to the vector $\mathbf{w}$ with initial point $(0,0,0)$ and terminal point $S-R=(2,1,0)-(1,-3,-2)=(2-1,1-(-3),0-(-2))=(1,4,2)$.
So $\overrightarrow{PQ}=\mathbf{v}=(1,4,2)$ and $\overrightarrow{RS}=\mathbf{w}=(1,4,2)$.
$\therefore \overrightarrow{PQ}=\overrightarrow{RS}$

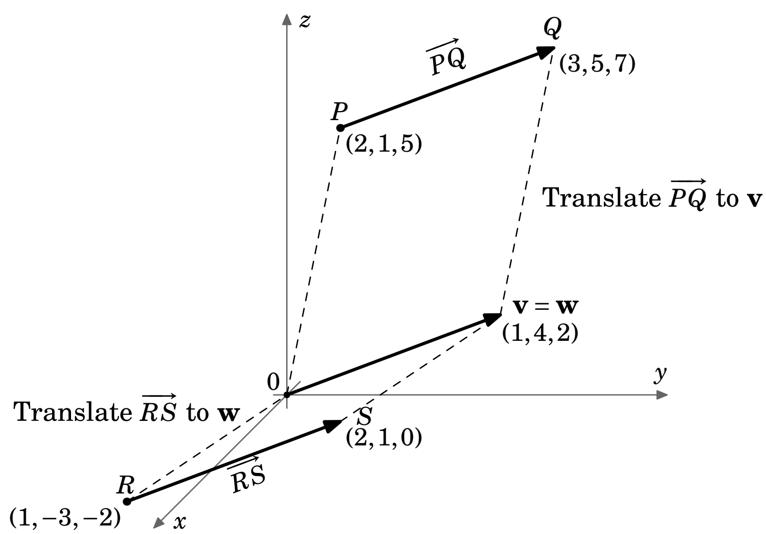


Figure 1.1.7

Recall the distance formula for points in the Euclidean plane:

For points $P=(x_1,y_1)$, $Q=(x_2,y_2)$ in $\mathbb{R}^2$, the distance d between P and Q is:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \tag{1.1}$$

By this formula, we have the following result:

For a vector $\overrightarrow{PQ}$ in $\mathbb{R}^2$ with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2)$, the magnitude of $\overrightarrow{PQ}$ is:

$$\|\overrightarrow{PQ}\|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \tag{1.2}$$

Finding the magnitude of a vector $\mathbf{v} = (a, b)$ in $\mathbb{R}^2$ is a special case of formula (1.2) with $P = (0,0)$ and $Q = (a, b)$:

For a vector $\mathbf{v} = (a, b)$ in $\mathbb{R}^2$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2} \tag{1.3}$$

To calculate the magnitude of vectors in $\mathbb{R}^3$, we need a distance formula for points in Euclidean space (we will postpone the proof until the next section):

Theorem 1.1. The distance d between points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1.4}$$

The proof will use the following result:

Theorem 1.2. For a vector $\mathbf{v} = (a, b, c)$ in $\mathbb{R}^3$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2} \tag{1.5}$$

Proof: There are four cases to consider:
Case 1: $a = b = c = 0$. Then $\mathbf{v} = \mathbf{0}$, so $\|\mathbf{v}\| = 0 = \sqrt{0^2 + 0^2 + 0^2} = \sqrt{a^2 + b^2 + c^2}$.

Case 2: exactly two of a, b, c *are* 0. Without loss of generality, we assume that $a = b = 0$ and $c \neq 0$ (the other two possibilities are handled in a similar manner). Then $\mathbf{v} = (0, 0, c)$, which is a vector of length $|c|$ along the z-axis. So $\|\mathbf{v}\| = |c| = \sqrt{c^2} = \sqrt{0^2 + 0^2 + c^2} = \sqrt{a^2 + b^2 + c^2}$.

Case 3: exactly one of a, b, c *is* 0. Without loss of generality, we assume that $a = 0$, $b \neq 0$ and $c \neq 0$ (the other two possibilities are handled in a similar manner). Then $\mathbf{v} = (0, b, c)$, which is a vector in the yz-plane, so by the Pythagorean Theorem we have $\|\mathbf{v}\| = \sqrt{b^2 + c^2} = \sqrt{0^2 + b^2 + c^2} = \sqrt{a^2 + b^2 + c^2}$.

Case 4: none of a, b, c *are* 0. Without loss of generality, we can assume that a, b, c are all positive (the other seven possibilities are handled in a similar manner). Consider the points $P = (0, 0, 0)$, $Q = (a, b, c)$, $R = (a, b, 0)$, and $S = (a, 0, 0)$, as shown in Figure 1.1.8. Applying the Pythagorean Theorem to the right triangle $\triangle PSR$ gives $|PR|^2 = a^2 + b^2$. A second application of the Pythagorean Theorem, this time to the right triangle $\triangle PQR$, gives $\|\mathbf{v}\| = |PQ| = \sqrt{|PR|^2 + |QR|^2} = \sqrt{a^2 + b^2 + c^2}$.
This proves the theorem. **QED**

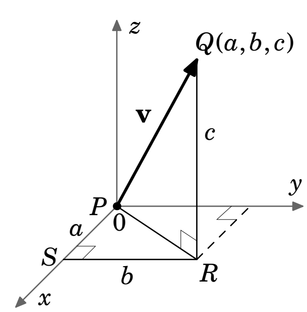


Figure 1.1.8

Example 1.3. Calculate the following:

(a) The magnitude of the vector $\overrightarrow{PQ}$ in $\mathbb{R}^2$ with $P = (-1, 2)$ and $Q = (5, 5)$.
Solution: By formula (1.2), $\|\overrightarrow{PQ}\| = \sqrt{(5-(-1))^2 + (5-2)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$.

(b) The magnitude of the vector $\mathbf{v} = (8, 3)$ in $\mathbb{R}^2$.
Solution: By formula (1.3), $\|\mathbf{v}\| = \sqrt{8^2 + 3^2} = \sqrt{73}$.

(c) The distance between the points $P = (2, -1, 4)$ and $Q = (4, 2, -3)$ in $\mathbb{R}^2$.
Solution: By formula (1.4), the distance $d = \sqrt{(4-2)^2 + (2-(-1))^2 + (-3-4)^2} = \sqrt{4+9+49} = \sqrt{62}$.

(d) The magnitude of the vector $\mathbf{v} = (5, 8, -2)$ in $\mathbb{R}^3$.
Solution: By formula (1.5), $\|\mathbf{v}\| = \sqrt{5^2 + 8^2 + (-2)^2} = \sqrt{25+64+4} = \sqrt{93}$.

Exercises

A

1. Calculate the magnitudes of the following vectors:
(a) $\mathbf{v} = (2, -1)$ (b) $\mathbf{v} = (2, -1, 0)$ (c) $\mathbf{v} = (3, 2, -2)$ (d) $\mathbf{v} = (0, 0, 1)$ (e) $\mathbf{v} = (6, 4, -4)$

2. For the points $P = (1, -1, 1)$, $Q = (2, -2, 2)$, $R = (2, 0, 1)$, $S = (3, -1, 2)$, does $\overrightarrow{PQ} = \overrightarrow{RS}$?

3. For the points $P = (0, 0, 0)$, $Q = (1, 3, 2)$, $R = (1, 0, 1)$, $S = (2, 3, 4)$, does $\overrightarrow{PQ} = \overrightarrow{RS}$?

B

4. Let $\mathbf{v} = (1, 0, 0)$ and $\mathbf{w} = (a, 0, 0)$ be vectors in $\mathbb{R}^3$. Show that $\|\mathbf{w}\| = |a|\,\|\mathbf{v}\|$.

5. Let $\mathbf{v} = (a, b, c)$ and $\mathbf{w} = (3a, 3b, 3c)$ be vectors in $\mathbb{R}^3$. Show that $\|\mathbf{w}\| = 3\,\|\mathbf{v}\|$.

C

6. Though we will see a simple proof of Theorem 1.1 in the next section, it is possible to prove it using methods similar to those in the proof of Theorem 1.2. Prove the special case of Theorem 1.1 where the points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ satisfy the following conditions:
$x_2 > x_1 > 0$, $y_2 > y_1 > 0$, and $z_2 > z_1 > 0$.
(*Hint: Think of Case 4 in the proof of Theorem 1.2, and consider Figure 1.1.9.*)

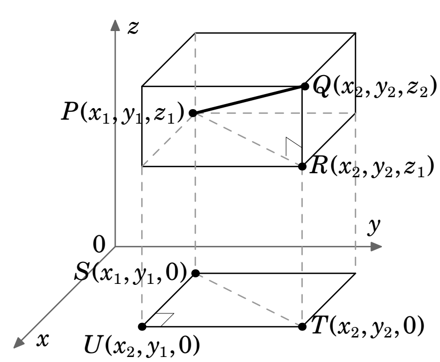


Figure 1.1.9

1.2 Vector Algebra

Now that we know what vectors are, we can start to perform some of the usual algebraic operations on them (e.g. addition, subtraction). Before doing that, we will introduce the notion of a *scalar*.

Definition 1.3. A **scalar** is a quantity that can be represented by a single number.

For our purposes, scalars will always be real numbers.[3] Examples of scalar quantities are mass, electric charge, and speed (not velocity).[4] We can now define *scalar multiplication* of a vector.

Definition 1.4. For a scalar k and a nonzero vector $\mathbf{v}$, the **scalar multiple** of $\mathbf{v}$ by k, denoted by $k\mathbf{v}$, is the vector whose magnitude is $|k|\,\|\mathbf{v}\|$, points in the same direction as $\mathbf{v}$ if $k > 0$, points in the opposite direction as $\mathbf{v}$ if $k < 0$, and is the zero vector $\mathbf{0}$ if $k = 0$. For the zero vector $\mathbf{0}$, we define $k\mathbf{0} = \mathbf{0}$ for any scalar k.

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are **parallel** (denoted by $\mathbf{v} \parallel \mathbf{w}$) if one is a scalar multiple of the other. You can think of scalar multiplication of a vector as stretching or shrinking the vector, and as flipping the vector in the opposite direction if the scalar is a negative number (see Figure 1.2.1).

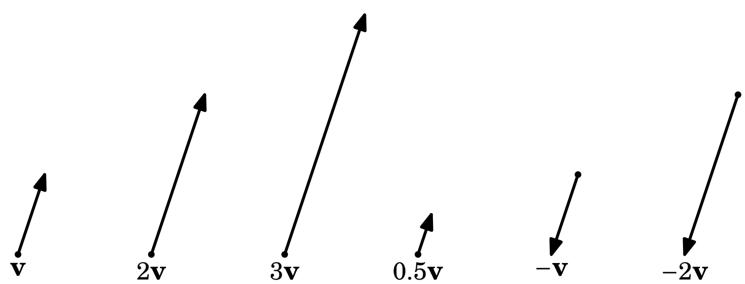


Figure 1.2.1

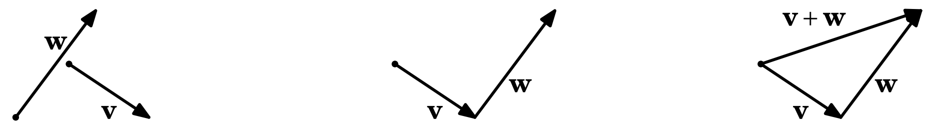
Recall that **translating** a nonzero vector means that the initial point of the vector is changed but the magnitude and direction are preserved. We are now ready to define the *sum* of two vectors.

Definition 1.5. The **sum** of vectors $\mathbf{v}$ and $\mathbf{w}$, denoted by $\mathbf{v}+\mathbf{w}$, is obtained by translating $\mathbf{w}$ so that its initial point is at the terminal point of $\mathbf{v}$; the initial point of $\mathbf{v}+\mathbf{w}$ is the initial point of $\mathbf{v}$, and its terminal point is the new terminal point of $\mathbf{w}$.

[3] The term *scalar* was invented by 19$^{\text{th}}$ century Irish mathematician, physicist and astronomer William Rowan Hamilton, to convey the sense of something that could be represented by a point on a scale or graduated ruler. The word vector comes from Latin, where it means "carrier".

[4] An alternate definition of scalars and vectors, used in physics, is that under certain types of coordinate transformations (e.g. rotations), a quantity that is not affected is a scalar, while a quantity that is affected (in a certain way) is a vector. See MARION for details.

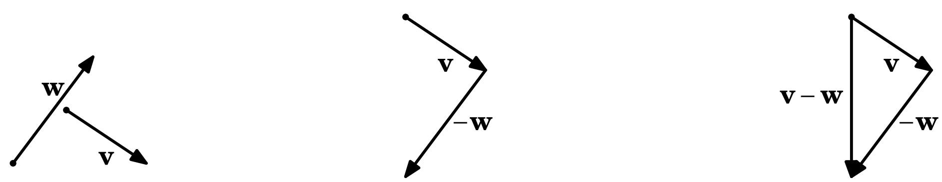
Intuitively, adding **w** to **v** means tacking on **w** to the end of **v** (see Figure 1.2.2).



(a) Vectors **v** and **w** **(b)** Translate **w** to the end of **v** **(c)** The sum $\mathbf{v}+\mathbf{w}$

Figure 1.2.2 Adding vectors **v** and **w**

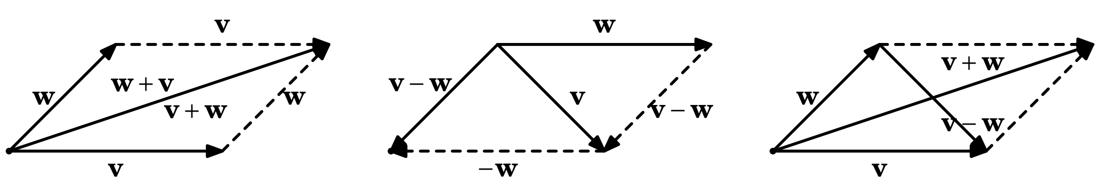
Notice that our definition is valid for the zero vector (which is just a point, and hence can be translated), and so we see that $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$ for any vector **v**. In particular, $\mathbf{0}+\mathbf{0}=\mathbf{0}$. Also, it is easy to see that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$, as we would expect. In general, since the scalar multiple $-\mathbf{v}=-1\,\mathbf{v}$ is a well-defined vector, we can define **vector subtraction** as follows: $\mathbf{v}-\mathbf{w}=\mathbf{v}+(-\mathbf{w})$. See Figure 1.2.3.



(a) Vectors **v** and **w** **(b)** Translate $-\mathbf{w}$ to the end of **v** **(c)** The difference $\mathbf{v}-\mathbf{w}$

Figure 1.2.3 Subtracting vectors **v** and **w**

Figure 1.2.4 shows the use of "geometric proofs" of various laws of vector algebra, that is, it uses laws from elementary geometry to prove statements about vectors. For example, (a) shows that $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ for any vectors **v**, **w**. And (c) shows how you can think of $\mathbf{v}-\mathbf{w}$ as the vector that is tacked on to the end of **w** to add up to **v**.



(a) Add vectors **(b)** Subtract vectors **(c)** Combined add/subtract

Figure 1.2.4 "Geometric" vector algebra

Notice that we have temporarily abandoned the practice of starting vectors at the origin. In fact, we have not even mentioned coordinates in this section so far. Since we will deal mostly with Cartesian coordinates in this book, the following two theorems are useful for performing vector algebra on vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ starting at the origin.

Theorem 1.3. Let $\mathbf{v} = (v_1, v_2)$, $\mathbf{w} = (w_1, w_2)$ be vectors in $\mathbb{R}^2$, and let k be a scalar. Then
(a) $k\mathbf{v} = (kv_1, kv_2)$
(b) $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)$

Proof: (a) Without loss of generality, we assume that $v_1, v_2 > 0$ (the other possibilities are handled in a similar manner). If $k = 0$ then $k\mathbf{v} = 0\mathbf{v} = \mathbf{0} = (0,0) = (0v_1, 0v_2) = (kv_1, kv_2)$, which is what we needed to show. If $k \neq 0$, then (kv_1, kv_2) lies on a line with slope $\frac{kv_2}{kv_1} = \frac{v_2}{v_1}$, which is the same as the slope of the line on which $\mathbf{v}$ (and hence $k\mathbf{v}$) lies, and (kv_1, kv_2) points in the same direction on that line as $k\mathbf{v}$. Also, by formula (1.3) the magnitude of (kv_1, kv_2) is $\sqrt{(kv_1)^2 + (kv_2)^2} = \sqrt{k^2v_1^2 + k^2v_2^2} = \sqrt{k^2(v_1^2 + v_2^2)} = |k|\sqrt{v_1^2 + v_2^2} = |k|\,\|\mathbf{v}\|$. So $k\mathbf{v}$ and (kv_1, kv_2) have the same magnitude and direction. This proves (a).

(b) Without loss of generality, we assume that $v_1, v_2, w_1, w_2 > 0$ (the other possibilities are handled in a similar manner). From Figure 1.2.5, we see that when translating $\mathbf{w}$ to start at the end of $\mathbf{v}$, the new terminal point of $\mathbf{w}$ is $(v_1 + w_1, v_2 + w_2)$, so by the definition of $\mathbf{v} + \mathbf{w}$ this must be the terminal point of $\mathbf{v} + \mathbf{w}$. This proves (b). **QED**

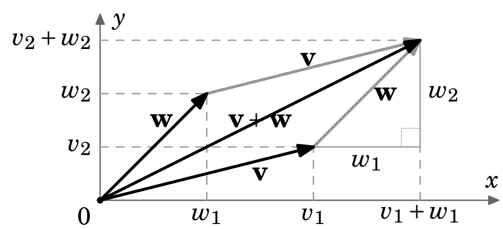


Figure 1.2.5

Theorem 1.4. Let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{R}^3$, let k be a scalar. Then
(a) $k\mathbf{v} = (kv_1, kv_2, kv_3)$
(b) $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$

The following theorem summarizes the basic laws of vector algebra.

Theorem 1.5. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and scalars k, l, we have

(a) $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	Commutative Law
(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative Law
(c) $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$	Additive Identity
(d) $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	Additive Inverse
(e) $k(l\mathbf{v}) = (kl)\mathbf{v}$	Associative Law
(f) $k(\mathbf{v} + \mathbf{w}) = k\mathbf{v} + k\mathbf{w}$	Distributive Law
(g) $(k + l)\mathbf{v} = k\mathbf{v} + l\mathbf{v}$	Distributive Law

Proof: (a) We already presented a geometric proof of this in Figure 1.2.4(a).

(b) To illustrate the difference between analytic proofs and geometric proofs in vector algebra, we will present both types here. For the analytic proof, we will use vectors in $\mathbb{R}^3$ (the proof for $\mathbb{R}^2$ is similar).

Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{R}^3$. Then

$$\begin{aligned}
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_1, u_2, u_3) + ((v_1, v_2, v_3) + (w_1, w_2, w_3)) \\
&= (u_1, u_2, u_3) + (v_1 + w_1, v_2 + w_2, v_3 + w_3) && \text{by Theorem 1.4(b)} \\
&= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3)) && \text{by Theorem 1.4(b)} \\
&= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3) && \text{by properties of real numbers} \\
&= (u_1 + v_1, u_2 + v_2, u_3 + v_3) + (w_1, w_2, w_3) && \text{by Theorem 1.4(b)} \\
&= (\mathbf{u} + \mathbf{v}) + \mathbf{w}
\end{aligned}$$

This completes the analytic proof of (b). Figure 1.2.6 provides the geometric proof.

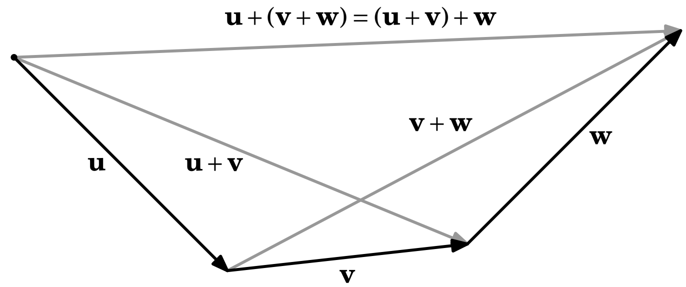


Figure 1.2.6 Associative Law for vector addition

(c) We already discussed this on p.10.

(d) We already discussed this on p.10.

(e) We will prove this for a vector $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ (the proof for $\mathbb{R}^2$ is similar):

$$\begin{aligned}
k(l\mathbf{v}) &= k(lv_1, lv_2, lv_3) && \text{by Theorem 1.4(a)} \\
&= (klv_1, klv_2, klv_3) && \text{by Theorem 1.4(a)} \\
&= (kl)(v_1, v_2, v_3) && \text{by Theorem 1.4(a)} \\
&= (kl)\mathbf{v}
\end{aligned}$$

(f) and (g): Left as exercises for the reader. **QED**

A **unit vector** is a vector with magnitude 1. Notice that for any nonzero vector $\mathbf{v}$, the vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector which points in the same direction as $\mathbf{v}$, since $\frac{1}{\|\mathbf{v}\|} > 0$ and $\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1$. Dividing a nonzero vector $\mathbf{v}$ by $\|\mathbf{v}\|$ is often called *normalizing* $\mathbf{v}$.

There are specific unit vectors which we will often use, called the **basis vectors**:
$\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, and $\mathbf{k} = (0,0,1)$ in $\mathbb{R}^3$; $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$ in $\mathbb{R}^2$.
These are useful for several reasons: they are mutually perpendicular, since they lie on distinct coordinate axes; they are all unit vectors: $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$; every vector can be written as a unique scalar combination of the basis vectors: $\mathbf{v} = (a,b) = a\,\mathbf{i} + b\,\mathbf{j}$ in $\mathbb{R}^2$, $\mathbf{v} = (a,b,c) = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ in $\mathbb{R}^3$. See Figure 1.2.7.

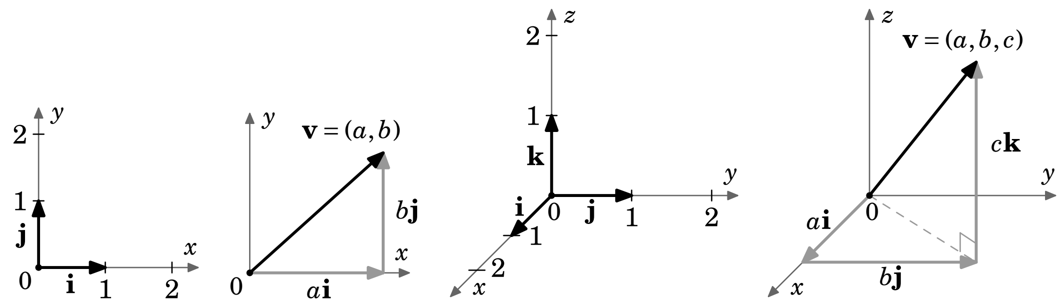


Figure 1.2.7 Basis vectors in different dimensions

When a vector $\mathbf{v} = (a,b,c)$ is written as $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$, we say that $\mathbf{v}$ is in *component form*, and that a, b, and c are the ***i**, **j**, and **k** components*, respectively, of $\mathbf{v}$. We have:

$$\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k},\, k \text{ a scalar} \Longrightarrow k\mathbf{v} = kv_1\,\mathbf{i} + kv_2\,\mathbf{j} + kv_3\,\mathbf{k}$$

$$\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k},\, \mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k} \Longrightarrow \mathbf{v} + \mathbf{w} = (v_1 + w_1)\mathbf{i} + (v_2 + w_2)\mathbf{j} + (v_3 + w_3)\mathbf{k}$$

$$\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k} \Longrightarrow \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Example 1.4. Let $\mathbf{v} = (2,1,-1)$ and $\mathbf{w} = (3,-4,2)$ in $\mathbb{R}^3$.

(a) Find $\mathbf{v} - \mathbf{w}$.
Solution: $\mathbf{v} - \mathbf{w} = (2-3, 1-(-4), -1-2) = (-1,5,-3)$

(b) Find $3\mathbf{v} + 2\mathbf{w}$.
Solution: $3\mathbf{v} + 2\mathbf{w} = (6,3,-3) + (6,-8,4) = (12,-5,1)$

(c) Write $\mathbf{v}$ and $\mathbf{w}$ in component form.
Solution: $\mathbf{v} = 2\,\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{w} = 3\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k}$

(d) Find the vector $\mathbf{u}$ such that $\mathbf{u} + \mathbf{v} = \mathbf{w}$.
Solution: By Theorem 1.5, $\mathbf{u} = \mathbf{w} - \mathbf{v} = -(\mathbf{v} - \mathbf{w}) = -(-1,5,-3) = (1,-5,3)$, by part(a).

(e) Find the vector $\mathbf{u}$ such that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.
Solution: By Theorem 1.5, $\mathbf{u} = -\mathbf{w} - \mathbf{v} = -(3,-4,2) - (2,1,-1) = (-5,3,-1)$.

(f) Find the vector $\mathbf{u}$ such that $2\mathbf{u} + \mathbf{i} - 2\mathbf{j} = \mathbf{k}$.
Solution: $2\mathbf{u} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Longrightarrow \mathbf{u} = -\frac{1}{2}\,\mathbf{i} + \mathbf{j} + \frac{1}{2}\,\mathbf{k}$

(g) Find the unit vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$.
Solution: $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2^2+1^2+(-1)^2}}(2,1,-1) = \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$

We can now easily prove Theorem 1.1 from the previous section. The distance d between two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ is the same as the length of the vector $\mathbf{w}-\mathbf{v}$, where the vectors $\mathbf{v}$ and $\mathbf{w}$ are defined as $\mathbf{v} = (x_1, y_1, z_1)$ and $\mathbf{w} = (x_2, y_2, z_2)$ (see Figure 1.2.8). So since $\mathbf{w}-\mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$, then $d = \|\mathbf{w}-\mathbf{v}\| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$ by Theorem 1.2.

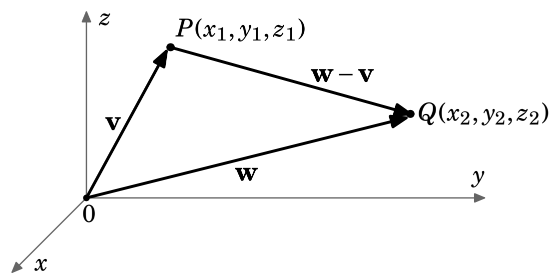


Figure 1.2.8 Proof of Theorem 1.2: $d = \|\mathbf{w}-\mathbf{v}\|$

Exercises

A

1. Let $\mathbf{v} = (-1, 5, -2)$ and $\mathbf{w} = (3, 1, 1)$.

(a) Find $\mathbf{v}-\mathbf{w}$. (b) Find $\mathbf{v}+\mathbf{w}$. (c) Find $\frac{\mathbf{v}}{\|\mathbf{v}\|}$. (d) Find $\left\|\frac{1}{2}(\mathbf{v}-\mathbf{w})\right\|$.

(e) Find $\left\|\frac{1}{2}(\mathbf{v}+\mathbf{w})\right\|$. (f) Find $-2\mathbf{v}+4\mathbf{w}$. (g) Find $\mathbf{v}-2\mathbf{w}$.

(h) Find the vector $\mathbf{u}$ such that $\mathbf{u}+\mathbf{v}+\mathbf{w} = \mathbf{i}$.

(i) Find the vector $\mathbf{u}$ such that $\mathbf{u}+\mathbf{v}+\mathbf{w} = 2\mathbf{j}+\mathbf{k}$.

(j) Is there a scalar m such that $m(\mathbf{v}+2\mathbf{w}) = \mathbf{k}$? If so, find it.

2. For the vectors $\mathbf{v}$ and $\mathbf{w}$ from Exercise 1, is $\|\mathbf{v}-\mathbf{w}\| = \|\mathbf{v}\| - \|\mathbf{w}\|$? If not, which quantity is larger?

3. For the vectors $\mathbf{v}$ and $\mathbf{w}$ from Exercise 1, is $\|\mathbf{v}+\mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$? If not, which quantity is larger?

B

4. Prove Theorem 1.5(f) for $\mathbb{R}^3$.

5. Prove Theorem 1.5(g) for $\mathbb{R}^3$.

C

6. We know that every vector in $\mathbb{R}^3$ can be written as a scalar combination of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Can every vector in $\mathbb{R}^3$ be written as a scalar combination of just $\mathbf{i}$ and $\mathbf{j}$, i.e. for any vector $\mathbf{v}$ in $\mathbb{R}^3$, are there scalars m, n such that $\mathbf{v} = m\,\mathbf{i} + n\,\mathbf{j}$? Justify your answer.

1.3 Dot Product

You may have noticed that while we did define multiplication of a vector by a scalar in the previous section on vector algebra, we did not define multiplication of a vector by a vector. We will now see one type of multiplication of vectors, called the *dot product*.

Definition 1.6. Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{R}^3$.
The **dot product** of $\mathbf{v}$ and $\mathbf{w}$, denoted by $\mathbf{v} \cdot \mathbf{w}$, is given by:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \tag{1.6}$$

Similarly, for vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ in $\mathbb{R}^2$, the dot product is:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 \tag{1.7}$$

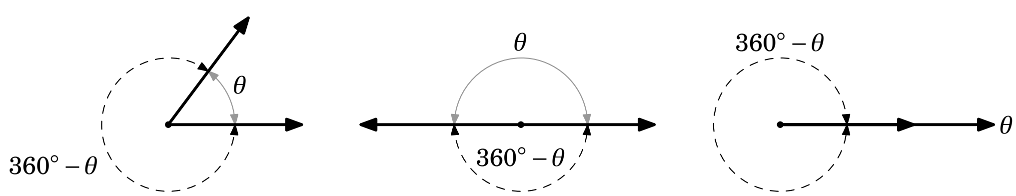
Notice that the dot product of two vectors is a scalar, not a vector. So the associative law that holds for multiplication of numbers and for addition of vectors (see Theorem 1.5(b),(e)), does *not* hold for the dot product of vectors. Why? Because for vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, the dot product $\mathbf{u} \cdot \mathbf{v}$ is a scalar, and so $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ is not defined since the left side of that dot product (the part in parentheses) is a scalar and not a vector.

For vectors $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$ in component form, the dot product is still $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$.

Also notice that we defined the dot product in an analytic way, i.e. by referencing vector coordinates. There is a geometric way of defining the dot product, which we will now develop as a consequence of the analytic definition.

Definition 1.7. The **angle** between two nonzero vectors with the same initial point is the smallest angle between them.

We do not define the angle between the zero vector and any other vector. Any two nonzero vectors with the same initial point have two angles between them: θ and $360° - \theta$. We will always choose the smallest nonnegative angle θ between them, so that $0° \le \theta \le 180°$. See Figure 1.3.1.



(a) $0° < \theta < 180°$ **(b)** $\theta = 180°$ **(c)** $\theta = 0°$

Figure 1.3.1 Angle between vectors

We can now take a more geometric view of the dot product by establishing a relationship between the dot product of two vectors and the angle between them.

Theorem 1.6. Let $\mathbf{v}$, $\mathbf{w}$ be nonzero vectors, and let θ be the angle between them. Then

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} \tag{1.8}$$

Proof: We will prove the theorem for vectors in $\mathbb{R}^3$ (the proof for $\mathbb{R}^2$ is similar). Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. By the Law of Cosines (see Figure 1.3.2), we have

$$\|\mathbf{v}-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta \tag{1.9}$$

(note that equation (1.9) holds even for the "degenerate" cases $\theta = 0°$ and $180°$).

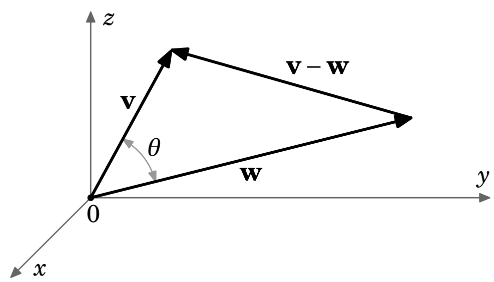


Figure 1.3.2

Since $\mathbf{v}-\mathbf{w} = (v_1 - w_1, v_2 - w_2, v_3 - w_3)$, expanding $\|\mathbf{v}-\mathbf{w}\|^2$ in equation (1.9) gives

$$\begin{aligned}
\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta &= (v_1-w_1)^2 + (v_2-w_2)^2 + (v_3-w_3)^2 \\
&= (v_1^2 - 2v_1w_1 + w_1^2) + (v_2^2 - 2v_2w_2 + w_2^2) + (v_3^2 - 2v_3w_3 + w_3^2) \\
&= (v_1^2 + v_2^2 + v_3^2) + (w_1^2 + w_2^2 + w_3^2) - 2(v_1w_1 + v_2w_2 + v_3w_3) \\
&= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2(\mathbf{v}\cdot\mathbf{w})\text{ , so} \\
-2\,\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta &= -2(\mathbf{v}\cdot\mathbf{w})\text{ , so since } \mathbf{v} \ne \mathbf{0} \text{ and } \mathbf{w} \ne \mathbf{0} \text{ then} \\
\cos\theta &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\text{ , since } \|\mathbf{v}\| > 0 \text{ and } \|\mathbf{w}\| > 0.
\end{aligned}$$

QED

Example 1.5. Find the angle θ between the vectors $\mathbf{v} = (2, 1, -1)$ and $\mathbf{w} = (3, -4, 1)$.

Solution: Since $\mathbf{v}\cdot\mathbf{w} = (2)(3) + (1)(-4) + (-1)(1) = 1$, $\|\mathbf{v}\| = \sqrt{6}$, and $\|\mathbf{w}\| = \sqrt{26}$, then

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \frac{1}{\sqrt{6}\sqrt{26}} = \frac{1}{2\sqrt{39}} \approx 0.08 \implies \theta = 85.41°$$

Two nonzero vectors are **perpendicular** if the angle between them is $90°$. Since $\cos 90° = 0$, we have the following important corollary to Theorem 1.6:

Corollary 1.7. Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are perpendicular if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

We will write $\mathbf{v} \perp \mathbf{w}$ to indicate that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

Since $\cos\theta > 0$ for $0° \le \theta < 90°$ and $\cos\theta < 0$ for $90° < \theta \le 180°$, we also have:

Corollary 1.8. If θ is the angle between nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, then

$$\mathbf{v}\cdot\mathbf{w} \ \text{ is } \begin{cases} >0 & \text{for } \ 0° \le \theta < 90° \\ 0 & \text{for } \ \theta = 90° \\ <0 & \text{for } \ 90° < \theta \le 180° \end{cases}$$

By Corollary 1.8, the dot product can be thought of as a way of telling if the angle between two vectors is acute, obtuse, or a right angle, depending on whether the dot product is positive, negative, or zero, respectively. See Figure 1.3.3.

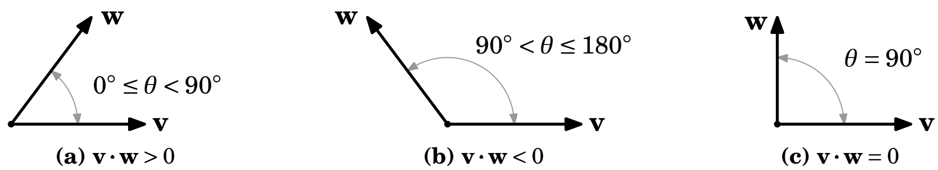


Figure 1.3.3 Sign of the dot product & angle between vectors

Example 1.6. Are the vectors $\mathbf{v} = (-1,5,-2)$ and $\mathbf{w} = (3,1,1)$ perpendicular?

Solution: Yes, $\mathbf{v} \perp \mathbf{w}$ since $\mathbf{v}\cdot\mathbf{w} = (-1)(3)+(5)(1)+(-2)(1) = 0$.

The following theorem summarizes the basic properties of the dot product.

Theorem 1.9. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and scalar k, we have

(a) $\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}$ Commutative Law

(b) $(k\mathbf{v})\cdot\mathbf{w} = \mathbf{v}\cdot(k\mathbf{w}) = k(\mathbf{v}\cdot\mathbf{w})$ Associative Law

(c) $\mathbf{v}\cdot\mathbf{0} = 0 = \mathbf{0}\cdot\mathbf{v}$

(d) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$ Distributive Law

(e) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$ Distributive Law

(f) $|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$ Cauchy-Schwarz Inequality[5]

Proof: The proofs of parts (a)-(e) are straightforward applications of the definition of the dot product, and are left to the reader as exercises. We will prove part (f).
(f) If either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$, then $\mathbf{v}\cdot\mathbf{w} = 0$ by part (c), and so the inequality holds trivially. So assume that $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors. Then by Theorem 1.6,

$$\begin{aligned} \mathbf{v}\cdot\mathbf{w} &= \cos\theta\,\|\mathbf{v}\|\,\|\mathbf{w}\|\ , \text{ so} \\ |\mathbf{v}\cdot\mathbf{w}| &= |\cos\theta|\,\|\mathbf{v}\|\,\|\mathbf{w}\|\ , \text{ so} \\ |\mathbf{v}\cdot\mathbf{w}| &\le \|\mathbf{v}\|\,\|\mathbf{w}\| \ \text{ since } |\cos\theta| \le 1. \end{aligned}$$ **QED**

[5]Also known as the Cauchy-Schwarz-Buniakovski Inequality.

Using Theorem 1.9, we see that if $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \cdot \mathbf{w} = 0$, then $\mathbf{u} \cdot (k\mathbf{v} + l\mathbf{w}) = k(\mathbf{u} \cdot \mathbf{v}) + l(\mathbf{u} \cdot \mathbf{w}) = k(0) + l(0) = 0$ for all scalars k, l. Thus, we have the following fact:

If $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$, then $\mathbf{u} \perp (k\mathbf{v} + l\mathbf{w})$ for all scalars k, l.

For vectors $\mathbf{v}$ and $\mathbf{w}$, the collection of all scalar combinations $k\mathbf{v} + l\mathbf{w}$ is called the **span** of $\mathbf{v}$ and $\mathbf{w}$. If nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel, then their span is a line; if they are not parallel, then their span is a plane. So what we showed above is that a vector which is perpendicular to two other vectors is also perpendicular to their span.

The dot product can be used to derive properties of the magnitudes of vectors, the most important of which is the *Triangle Inequality*, as given in the following theorem:

Theorem 1.10. For any vectors $\mathbf{v}$, $\mathbf{w}$, we have

(a) $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$

(b) $\|\mathbf{v} + \mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$ Triangle Inequality

(c) $\|\mathbf{v} - \mathbf{w}\| \ge \|\mathbf{v}\| - \|\mathbf{w}\|$

Proof: (a) Left as an exercise for the reader.

(b) By part (a) and Theorem 1.9, we have

$$\begin{aligned}
\|\mathbf{v} + \mathbf{w}\|^2 &= (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} \\
&= \|\mathbf{v}\|^2 + 2(\mathbf{v} \cdot \mathbf{w}) + \|\mathbf{w}\|^2 \text{ , so since } a \le |a| \text{ for any real number } a \text{, we have} \\
&\le \|\mathbf{v}\|^2 + 2\,|\mathbf{v} \cdot \mathbf{w}| + \|\mathbf{w}\|^2 \text{ , so by Theorem 1.9(f) we have} \\
&\le \|\mathbf{v}\|^2 + 2\,\|\mathbf{v}\|\,\|\mathbf{w}\| + \|\mathbf{w}\|^2 = (\|\mathbf{v}\| + \|\mathbf{w}\|)^2 \text{ and so}
\end{aligned}$$

$\|\mathbf{v} + \mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$ after taking square roots of both sides, which proves (b).

(c) Since $\mathbf{v} = \mathbf{w} + (\mathbf{v} - \mathbf{w})$, then $\|\mathbf{v}\| = \|\mathbf{w} + (\mathbf{v} - \mathbf{w})\| \le \|\mathbf{w}\| + \|\mathbf{v} - \mathbf{w}\|$ by the Triangle Inequality, so subtracting $\|\mathbf{w}\|$ from both sides gives $\|\mathbf{v}\| - \|\mathbf{w}\| \le \|\mathbf{v} - \mathbf{w}\|$. **QED**

The Triangle Inequality gets its name from the fact that in any triangle, no one side is longer than the sum of the lengths of the other two sides (see Figure 1.3.4). Another way of saying this is with the familiar statement "the shortest distance between two points is a straight line."

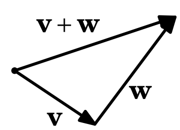


Figure 1.3.4

Exercises

A

1. Let $\mathbf{v} = (5, 1, -2)$ and $\mathbf{w} = (4, -4, 3)$. Calculate $\mathbf{v} \cdot \mathbf{w}$.

2. Let $\mathbf{v} = -3\,\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = 6\,\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. Calculate $\mathbf{v} \cdot \mathbf{w}$.

For Exercises 3-8, find the angle θ between the vectors $\mathbf{v}$ and $\mathbf{w}$.

3. $\mathbf{v} = (5,1,-2)$, $\mathbf{w} = (4,-4,3)$

4. $\mathbf{v} = (7,2,-10)$, $\mathbf{w} = (2,6,4)$

5. $\mathbf{v} = (2,1,4)$, $\mathbf{w} = (1,-2,0)$

6. $\mathbf{v} = (4,2,-1)$, $\mathbf{w} = (8,4,-2)$

7. $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{w} = -3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$

8. $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

9. Let $\mathbf{v} = (8,4,3)$ and $\mathbf{w} = (-2,1,4)$. Is $\mathbf{v} \perp \mathbf{w}$? Justify your answer.

10. Let $\mathbf{v} = (6,0,4)$ and $\mathbf{w} = (0,2,-1)$. Is $\mathbf{v} \perp \mathbf{w}$? Justify your answer.

11. For $\mathbf{v}$, $\mathbf{w}$ from Exercise 5, verify the Cauchy-Schwarz Inequality $|\mathbf{v} \cdot \mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$.

12. For $\mathbf{v}$, $\mathbf{w}$ from Exercise 6, verify the Cauchy-Schwarz Inequality $|\mathbf{v} \cdot \mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$.

13. For $\mathbf{v}$, $\mathbf{w}$ from Exercise 5, verify the Triangle Inequality $\|\mathbf{v} + \mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$.

14. For $\mathbf{v}$, $\mathbf{w}$ from Exercise 6, verify the Triangle Inequality $\|\mathbf{v} + \mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$.

B

Note: Consider only vectors in $\mathbb{R}^3$ for Exercises 15-25.

15. Prove Theorem 1.9(a).

16. Prove Theorem 1.9(b).

17. Prove Theorem 1.9(c).

18. Prove Theorem 1.9(d).

19. Prove Theorem 1.9(e).

20. Prove Theorem 1.10(a).

21. Prove or give a counterexample: If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

C

22. Prove or give a counterexample: If $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{v}$, then $\mathbf{w} = \mathbf{0}$.

1.4 Cross Product

In Section 1.3 we defined the dot product, which gave a way of multiplying two vectors. The resulting product, however, was a scalar, not a vector. In this section we will define a product of two vectors that does result in another vector. This product, called the *cross product*, is only defined for vectors in $\mathbb{R}^3$. The definition may appear strange and lacking motivation, but we will see the geometric basis for it shortly.

Definition 1.8. Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{R}^3$. The **cross product** of $\mathbf{v}$ and $\mathbf{w}$, denoted by $\mathbf{v} \times \mathbf{w}$, is the vector in $\mathbb{R}^3$ given by:

$$\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \tag{1.10}$$

Example 1.7. Find $\mathbf{i} \times \mathbf{j}$.

Solution: Since $\mathbf{i} = (1,0,0)$ and $\mathbf{j} = (0,1,0)$, then

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= ((0)(0) - (0)(1), (0)(0) - (1)(0), (1)(1) - (0)(0))\\ &= (0,0,1)\\ &= \mathbf{k}\end{aligned}$$

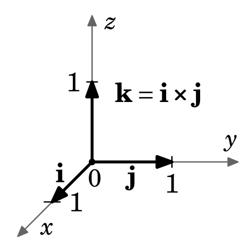


Figure 1.4.1

Similarly it can be shown that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

In the above example, the cross product of the given vectors was perpendicular to both those vectors. It turns out that this will always be the case.

Theorem 1.11. If the cross product $\mathbf{v} \times \mathbf{w}$ of two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ is also a nonzero vector, then it is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

Proof: We will show that $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} = 0$:

$$\begin{aligned}(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} &= (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \cdot (v_1, v_2, v_3)\\ &= v_2w_3v_1 - v_3w_2v_1 + v_3w_1v_2 - v_1w_3v_2 + v_1w_2v_3 - v_2w_1v_3\\ &= v_1v_2w_3 - v_1v_2w_3 + w_1v_2v_3 - w_1v_2v_3 + v_1w_2v_3 - v_1w_2v_3\\ &= 0 \text{ , after rearranging the terms.}\end{aligned}$$

$\therefore \mathbf{v} \times \mathbf{w} \perp \mathbf{v}$ by Corollary 1.7.
The proof that $\mathbf{v} \times \mathbf{w} \perp \mathbf{w}$ is similar. **QED**

As a consequence of the above theorem and Theorem 1.9, we have the following:

Corollary 1.12. If the cross product $\mathbf{v} \times \mathbf{w}$ of two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ is also a nonzero vector, then it is perpendicular to the span of $\mathbf{v}$ and $\mathbf{w}$.

The span of any two nonzero, nonparallel vectors $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$ is a plane P, so the above corollary shows that $\mathbf{v}\times\mathbf{w}$ is perpendicular to that plane. As shown in Figure 1.4.2, there are two possible directions for $\mathbf{v}\times\mathbf{w}$, one the opposite of the other. It turns out (see Appendix B) that the direction of $\mathbf{v}\times\mathbf{w}$ is given by the *right-hand rule*, that is, the vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v}\times\mathbf{w}$ form a right-handed system. Recall from Section 1.1 that this means that you can point your thumb upwards in the direction of $\mathbf{v}\times\mathbf{w}$ while rotating $\mathbf{v}$ towards $\mathbf{w}$ with the remaining four fingers.

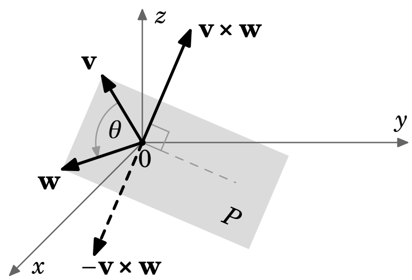


Figure 1.4.2 Direction of $\mathbf{v}\times\mathbf{w}$

We will now derive a formula for the magnitude of $\mathbf{v}\times\mathbf{w}$, for nonzero vectors $\mathbf{v}$, $\mathbf{w}$:

$$\begin{aligned}\|\mathbf{v}\times\mathbf{w}\|^2 &= (v_2w_3-v_3w_2)^2+(v_3w_1-v_1w_3)^2+(v_1w_2-v_2w_1)^2\\ &= v_2^2w_3^2-2v_2w_2v_3w_3+v_3^2w_2^2+v_3^2w_1^2-2v_1w_1v_3w_3+v_1^2w_3^2+v_1^2w_2^2-2v_1w_1v_2w_2+v_2^2w_1^2\\ &= v_1^2(w_2^2+w_3^2)+v_2^2(w_1^2+w_3^2)+v_3^2(w_1^2+w_2^2)-2(v_1w_1v_2w_2+v_1w_1v_3w_3+v_2w_2v_3w_3)\end{aligned}$$

and now adding and subtracting $v_1^2w_1^2$, $v_2^2w_2^2$, and $v_3^2w_3^2$ on the right side gives

$$\begin{aligned}&= v_1^2(w_1^2+w_2^2+w_3^2)+v_2^2(w_1^2+w_2^2+w_3^2)+v_3^2(w_1^2+w_2^2+w_3^2)\\ &\quad -(v_1^2w_1^2+v_2^2w_2^2+v_3^2w_3^2+2(v_1w_1v_2w_2+v_1w_1v_3w_3+v_2w_2v_3w_3))\\ &= (v_1^2+v_2^2+v_3^2)(w_1^2+w_2^2+w_3^2)\\ &\quad -((v_1w_1)^2+(v_2w_2)^2+(v_3w_3)^2+2(v_1w_1)(v_2w_2)+2(v_1w_1)(v_3w_3)+2(v_2w_2)(v_3w_3))\end{aligned}$$

so using $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$ for the subtracted term gives

$$\begin{aligned}&= (v_1^2+v_2^2+v_3^2)(w_1^2+w_2^2+w_3^2)-(v_1w_1+v_2w_2+v_3w_3)^2\\ &= \|\mathbf{v}\|^2\,\|\mathbf{w}\|^2-(\mathbf{v}\cdot\mathbf{w})^2\\ &= \|\mathbf{v}\|^2\,\|\mathbf{w}\|^2\left(1-\frac{(\mathbf{v}\cdot\mathbf{w})^2}{\|\mathbf{v}\|^2\,\|\mathbf{w}\|^2}\right)\text{, since } \|\mathbf{v}\|>0 \text{ and } \|\mathbf{w}\|>0\text{, so by Theorem 1.6}\\ &= \|\mathbf{v}\|^2\,\|\mathbf{w}\|^2(1-\cos^2\theta)\text{, where } \theta \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{w}\text{, so}\end{aligned}$$

$\|\mathbf{v}\times\mathbf{w}\|^2=\|\mathbf{v}\|^2\,\|\mathbf{w}\|^2\sin^2\theta$, and since $0^\circ\le\theta\le180^\circ$, then $\sin\theta\ge0$, so we have:

If θ is the angle between nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, then

$$\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\,\|\mathbf{w}\|\,\sin\theta \tag{1.11}$$

It may seem strange to bother with the above formula, when the magnitude of the cross product can be calculated directly, like for any other vector. The formula is more useful for its applications in geometry, as in the following example.

Example 1.8. Let $\triangle PQR$ and $PQRS$ be a triangle and parallelogram, respectively, as shown in Figure 1.4.3.

Figure 1.4.3

Think of the triangle as existing in $\mathbb{R}^3$, and identify the sides QR and QP with vectors $\mathbf{v}$ and $\mathbf{w}$, respectively, in $\mathbb{R}^3$. Let θ be the angle between $\mathbf{v}$ and $\mathbf{w}$. The area A_{PQR} of $\triangle PQR$ is $\frac{1}{2}bh$, where b is the base of the triangle and h is the height. So we see that

$$b = \|\mathbf{v}\| \quad \text{and} \quad h = \|\mathbf{w}\|\sin\theta$$

$$\begin{aligned} A_{PQR} &= \frac{1}{2}\,\|\mathbf{v}\|\,\|\mathbf{w}\|\,\sin\theta \\ &= \frac{1}{2}\,\|\mathbf{v}\times\mathbf{w}\| \end{aligned}$$

So since the area A_{PQRS} of the parallelogram $PQRS$ is twice the area of the triangle $\triangle PQR$, then

$$A_{PQRS} = \|\mathbf{v}\|\,\|\mathbf{w}\|\,\sin\theta$$

By the discussion in Example 1.8, we have proved the following theorem:

Theorem 1.13. Area of triangles and parallelograms

(a) The area A of a triangle with adjacent sides $\mathbf{v}$, $\mathbf{w}$ (as vectors in $\mathbb{R}^3$) is:

$$A = \frac{1}{2}\,\|\mathbf{v}\times\mathbf{w}\|$$

(b) The area A of a parallelogram with adjacent sides $\mathbf{v}$, $\mathbf{w}$ (as vectors in $\mathbb{R}^3$) is:

$$A = \|\mathbf{v}\times\mathbf{w}\|$$

It may seem at first glance that since the formulas derived in Example 1.8 were for the adjacent sides QP and QR only, then the more general statements in Theorem 1.13 that the formulas hold for *any* adjacent sides are not justified. We would get a different *formula* for the area if we had picked PQ and PR as the adjacent sides, but it can be shown (see Exercise 26) that the different formulas would yield the same value, so the choice of adjacent sides indeed does not matter, and Theorem 1.13 is valid.

Theorem 1.13 makes it simpler to calculate the area of a triangle in 3-dimensional space than by using traditional geometric methods.

Example 1.9. Calculate the area of the triangle $\triangle PQR$, where $P = (2,4,-7)$, $Q = (3,7,18)$, and $R = (-5,12,8)$.

Solution: Let $\mathbf{v} = \overrightarrow{PQ}$ and $\mathbf{w} = \overrightarrow{PR}$, as in Figure 1.4.4. Then $\mathbf{v} = (3,7,18)-(2,4,-7) = (1,3,25)$ and $\mathbf{w} = (-5,12,8)-(2,4,-7) = (-7,8,15)$, so the area A of the triangle $\triangle PQR$ is

$$\begin{aligned}
A &= \frac{1}{2}\,\|\mathbf{v}\times\mathbf{w}\| = \frac{1}{2}\,\|(1,3,25)\times(-7,8,15)\| \\
&= \frac{1}{2}\,\big\|((3)(15)-(25)(8),(25)(-7)-(1)(15),(1)(8)-(3)(-7))\big\| \\
&= \frac{1}{2}\,\big\|(-155,-190,29)\big\| \\
&= \frac{1}{2}\,\sqrt{(-155)^2+(-190)^2+29^2} = \frac{1}{2}\,\sqrt{60966} \\
A &\approx 123.46
\end{aligned}$$

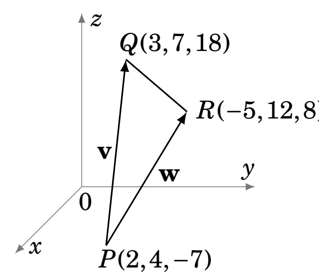


Figure 1.4.4

Example 1.10. Calculate the area of the parallelogram $PQRS$, where $P = (1,1)$, $Q = (2,3)$, $R = (5,4)$, and $S = (4,2)$.

Solution: Let $\mathbf{v} = \overrightarrow{SP}$ and $\mathbf{w} = \overrightarrow{SR}$, as in Figure 1.4.5. Then $\mathbf{v} = (1,1)-(4,2) = (-3,-1)$ and $\mathbf{w} = (5,4)-(4,2) = (1,2)$. But these are vectors in $\mathbb{R}^2$, and the cross product is only defined for vectors in $\mathbb{R}^3$. However, $\mathbb{R}^2$ can be thought of as the subset of $\mathbb{R}^3$ such that the z-coordinate is always 0. So we can write $\mathbf{v} = (-3,-1,0)$ and $\mathbf{w} = (1,2,0)$. Then the area A of $PQRS$ is

$$\begin{aligned}
A &= \|\mathbf{v}\times\mathbf{w}\| = \big\|(-3,-1,0)\times(1,2,0)\big\| \\
&= \big\|((-1)(0)-(0)(2),(0)(1)-(-3)(0),(-3)(2)-(-1)(1))\big\| \\
&= \big\|(0,0,-5)\big\| \\
A &= 5
\end{aligned}$$

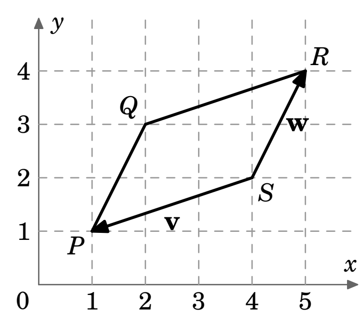


Figure 1.4.5

The following theorem summarizes the basic properties of the cross product.

Theorem 1.14. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$, and scalar k, we have

(a) $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$ Anticommutative Law

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ Distributive Law

(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$ Distributive Law

(d) $(k\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (k\mathbf{w}) = k(\mathbf{v} \times \mathbf{w})$ Associative Law

(e) $\mathbf{v} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{v}$

(f) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

(g) $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v} \parallel \mathbf{w}$

Proof: The proofs of properties (b)-(f) are straightforward. We will prove parts (a) and (g) and leave the rest to the reader as exercises.

(a) By the definition of the cross product and scalar multiplication, we have:

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &= (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1)\\ &= -(v_3w_2 - v_2w_3, v_1w_3 - v_3w_1, v_2w_1 - v_1w_2)\\ &= -(w_2v_3 - w_3v_2, w_3v_1 - w_1v_3, w_1v_2 - w_2v_1)\\ &= -\mathbf{w} \times \mathbf{v}\end{aligned}$$

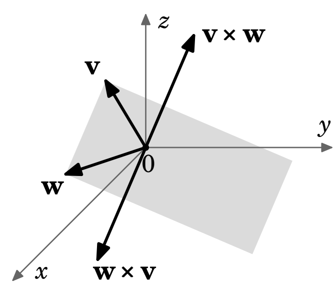


Figure 1.4.6

Note that this says that $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{v}$ have the same magnitude but opposite direction (see Figure 1.4.6).

(g) If either $\mathbf{v}$ or $\mathbf{w}$ is $\mathbf{0}$ then $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ by part (e), and either $\mathbf{v} = \mathbf{0} = 0\mathbf{w}$ or $\mathbf{w} = \mathbf{0} = 0\mathbf{v}$, so $\mathbf{v}$ and $\mathbf{w}$ are scalar multiples, i.e. they are parallel.

If both $\mathbf{v}$ and $\mathbf{w}$ are nonzero, and θ is the angle between them, then by formula (1.11), $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\|\mathbf{v}\|\,\|\mathbf{w}\| \sin\theta = 0$, which is true if and only if $\sin\theta = 0$ (since $\|\mathbf{v}\| > 0$ and $\|\mathbf{w}\| > 0$). So since $0° \le \theta \le 180°$, then $\sin\theta = 0$ if and only if $\theta = 0°$ or $180°$. But the angle between $\mathbf{v}$ and $\mathbf{w}$ is $0°$ or $180°$ if and only if $\mathbf{v} \parallel \mathbf{w}$. **QED**

Example 1.11. Adding to Example 1.7, we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

Recall from geometry that a *parallelepiped* is a 3-dimensional solid with 6 faces, all of which are parallelograms.[6]

[6]An equivalent definition of a parallelepiped is: the collection of all scalar combinations $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ of some vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^3$, where $0 \le k_1, k_2, k_3 \le 1$.

Example 1.12. *Volume of a parallelepiped:* Let the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$ represent adjacent sides of a parallelepiped P, with $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ forming a right-handed system, as in Figure 1.4.7. Show that the volume of P is the *scalar triple product* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$.

Solution: Recall that the volume vol(P) of a parallelepiped P is the area A of the base parallelogram times the height h. By Theorem 1.13(b), the area A of the base parallelogram is $\|\mathbf{v}\times\mathbf{w}\|$. And we can see that since $\mathbf{v}\times\mathbf{w}$ is perpendicular to the base parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$, then the height h is $\|\mathbf{u}\|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$. By Theorem 1.6 we know that

$$\cos\theta = \frac{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}{\|\mathbf{u}\|\,\|\mathbf{v}\times\mathbf{w}\|}.\quad \text{Hence,}$$

$$\begin{aligned}\mathrm{vol}(P) &= A\,h\\ &= \|\mathbf{v}\times\mathbf{w}\|\,\frac{\|\mathbf{u}\|\,\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}{\|\mathbf{u}\|\,\|\mathbf{v}\times\mathbf{w}\|}\\ &= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\end{aligned}$$

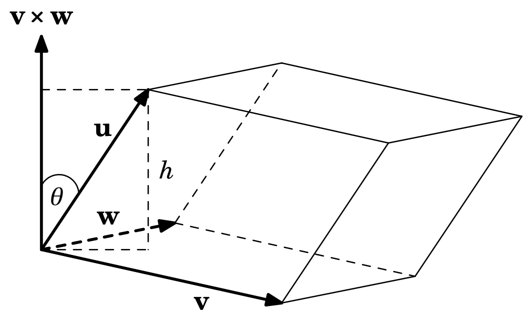


Figure 1.4.7 Parallelepiped P

In Example 1.12 the height h of the parallelepiped is $\|\mathbf{u}\|\cos\theta$, and not $-\|\mathbf{u}\|\cos\theta$, because the vector $\mathbf{u}$ is on the same side of the base parallelogram's plane as the vector $\mathbf{v}\times\mathbf{w}$ (so that $\cos\theta > 0$). Since the volume is the same no matter which base and height we use, then repeating the same steps using the base determined by $\mathbf{u}$ and $\mathbf{v}$ (since $\mathbf{w}$ is on the same side of that base's plane as $\mathbf{u}\times\mathbf{v}$), the volume is $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$. Repeating this with the base determined by $\mathbf{w}$ and $\mathbf{u}$, we have the following result:

For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$,

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) \tag{1.12}$$

(Note that the equalities hold trivially if any of the vectors are $\mathbf{0}$.)

Since $\mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v}$ for any vectors $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$, then picking the wrong order for the three adjacent sides in the scalar triple product in formula (1.12) will give you the negative of the volume of the parallelepiped. So taking the absolute value of the scalar triple product for *any* order of the three adjacent sides will *always* give the volume:

Theorem 1.15. If vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$ represent any three adjacent sides of a parallelepiped, then the volume of the parallelepiped is $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$.

Another type of triple product is the *vector triple product* $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$. The proof of the following theorem is left as an exercise for the reader:

Theorem 1.16. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \tag{1.13}$$

An examination of the formula in Theorem 1.16 gives some idea of the geometry of the vector triple product. By the right side of formula (1.13), we see that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is a scalar combination of $\mathbf{v}$ and $\mathbf{w}$, and hence lies in the plane containing $\mathbf{v}$ and $\mathbf{w}$ (i.e. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, $\mathbf{v}$ and $\mathbf{w}$ are **coplanar**). This makes sense since, by Theorem 1.11, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$. In particular, being perpendicular to $\mathbf{v} \times \mathbf{w}$ means that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane containing $\mathbf{v}$ and $\mathbf{w}$, since that plane is itself perpendicular to $\mathbf{v} \times \mathbf{w}$. But then how is $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ also perpendicular to $\mathbf{u}$, which could be any vector? The following example may help to see how this works.

Example 1.13. Find $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ for $\mathbf{u} = (1,2,4)$, $\mathbf{v} = (2,2,0)$, $\mathbf{w} = (1,3,0)$.

Solution: Since $\mathbf{u} \cdot \mathbf{v} = 6$ and $\mathbf{u} \cdot \mathbf{w} = 7$, then

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \\
&= 7(2,2,0) - 6(1,3,0) = (14,14,0) - (6,18,0) \\
&= (8,-4,0)
\end{aligned}$$

Note that $\mathbf{v}$ and $\mathbf{w}$ lie in the xy-plane, and that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ also lies in that plane. Also, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w} = (0,0,4)$ (see Figure 1.4.8).

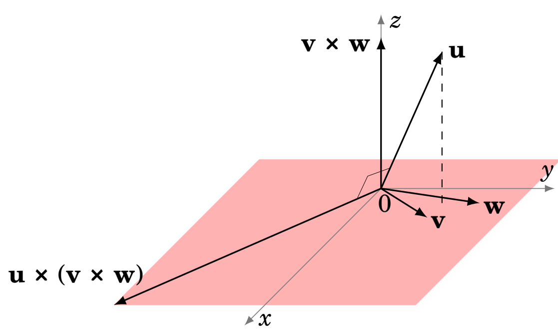


Figure 1.4.8

For vectors $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$ in component form, the cross product is written as: $\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$. It is often easier to use the component form for the cross product, because it can be represented as a *determinant*. We will not go too deeply into the theory of determinants[7]; we will just cover what is essential for our purposes.

A **2 × 2 matrix** is an array of two rows and two columns of scalars, written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are scalars. The **determinant** of such a matrix, written as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{or} \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

is the scalar defined by the following formula:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

It may help to remember this formula as being the product of the scalars on the downward diagonal minus the product of the scalars on the upward diagonal.

Example 1.14.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = 4 - 6 = -2$$

A **3 × 3 matrix** is an array of three rows and three columns of scalars, written as

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

and its determinant is given by the formula:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{1.14}$$

One way to remember the above formula is the following: multiply each scalar in the first row by the determinant of the 2×2 matrix that remains after removing the row and column that contain that scalar, then sum those products up, putting alternating plus and minus signs in front of each (starting with a plus).

Example 1.15.

$$\begin{vmatrix} 1 & 0 & 2 \\ 4 & -1 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 1 \begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix} - 0 \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix} = 1(-2-0) - 0(8-3) + 2(0+1) = 0$$

We defined the determinant as a scalar, derived from algebraic operations on scalar entries in a matrix. However, if we put three *vectors* in the first row of a 3×3 matrix, then the definition still makes sense, since we would be performing scalar multiplication on those three vectors (they would be multiplied by the 2×2 scalar determinants as before). This gives us a determinant that is now a vector, and lets us write the cross product of $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ and $\mathbf{w} = w_1\,\mathbf{i} + w_2\,\mathbf{j} + w_3\,\mathbf{k}$ as a determinant:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\mathbf{k}$$

$$= (v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$$

Example 1.16. Let $\mathbf{v} = 4\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\,\mathbf{k}$. Then

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 3 \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{k} = -2\,\mathbf{i} - 5\,\mathbf{j} + \mathbf{k}$$

The scalar triple product can also be written as a determinant. In fact, by Example 1.12, the following theorem provides an alternate definition of the determinant of a 3×3 matrix as the volume of a parallelepiped whose adjacent sides are the rows of the matrix and form a right-handed system (a left-handed system would give the negative volume).

Theorem 1.17. For any vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$ in $\mathbb{R}^3$:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{1.15}$$

Example 1.17. Find the volume of the parallelepiped with adjacent sides $\mathbf{u} = (2, 1, 3)$, $\mathbf{v} = (-1, 3, 2)$, $\mathbf{w} = (1, 1, -2)$ (see Figure 1.4.9).

Solution: By Theorem 1.15, the volume $\text{vol}(P)$ of the parallelepiped P is the absolute value of the scalar triple product of the three adjacent sides (in any order). By Theorem 1.17,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 2(-8) - 1(0) + 3(-4) = -28, \text{ so}$$

$$\text{vol}(P) = |-28| = 28.$$

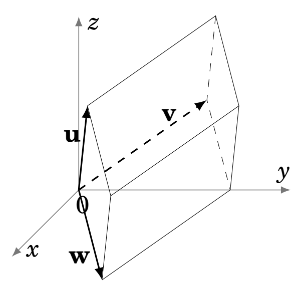


Figure 1.4.9 P

Interchanging the dot and cross products can be useful in proving vector identities:

Example 1.18. Prove: $(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{z}) = \begin{vmatrix}\mathbf{u}\cdot\mathbf{w} & \mathbf{u}\cdot\mathbf{z}\\ \mathbf{v}\cdot\mathbf{w} & \mathbf{v}\cdot\mathbf{z}\end{vmatrix}$ for all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$ in $\mathbb{R}^3$.

Solution: Let $\mathbf{x} = \mathbf{u}\times\mathbf{v}$. Then

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{z}) &= \mathbf{x}\cdot(\mathbf{w}\times\mathbf{z})\\ &= \mathbf{w}\cdot(\mathbf{z}\times\mathbf{x}) \quad \text{(by formula (1.12))}\\ &= \mathbf{w}\cdot(\mathbf{z}\times(\mathbf{u}\times\mathbf{v}))\\ &= \mathbf{w}\cdot((\mathbf{z}\cdot\mathbf{v})\mathbf{u}-(\mathbf{z}\cdot\mathbf{u})\mathbf{v}) \quad \text{(by Theorem 1.16)}\\ &= (\mathbf{z}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{u})-(\mathbf{z}\cdot\mathbf{u})(\mathbf{w}\cdot\mathbf{v})\\ &= (\mathbf{u}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{z})-(\mathbf{u}\cdot\mathbf{z})(\mathbf{v}\cdot\mathbf{w}) \quad \text{(by commutativity of the dot product).}\\ &= \begin{vmatrix}\mathbf{u}\cdot\mathbf{w} & \mathbf{u}\cdot\mathbf{z}\\ \mathbf{v}\cdot\mathbf{w} & \mathbf{v}\cdot\mathbf{z}\end{vmatrix}\end{aligned}$$

Exercises

A

For Exercises 1-6, calculate $\mathbf{v}\times\mathbf{w}$.

1. $\mathbf{v} = (5,1,-2)$, $\mathbf{w} = (4,-4,3)$

2. $\mathbf{v} = (7,2,-10)$, $\mathbf{w} = (2,6,4)$

3. $\mathbf{v} = (2,1,4)$, $\mathbf{w} = (1,-2,0)$

4. $\mathbf{v} = (1,3,2)$, $\mathbf{w} = (7,2,-10)$

5. $\mathbf{v} = -\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{w} = -3\mathbf{i}+6\mathbf{j}+3\mathbf{k}$

6. $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = 3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$

For Exercises 7-8, calculate the area of the triangle $\triangle PQR$.

7. $P = (5,1,-2)$, $Q = (4,-4,3)$, $R = (2,4,0)$

8. $P = (4,0,2)$, $Q = (2,1,5)$, $R = (-1,0,-1)$

For Exercises 9-10, calculate the area of the parallelogram $PQRS$.

9. $P = (2,1,3)$, $Q = (1,4,5)$, $R = (2,5,3)$, $S = (3,2,1)$

10. $P = (-2,-2)$, $Q = (1,4)$, $R = (6,6)$, $S = (3,0)$

For Exercises 11-12, find the volume of the parallelepiped with adjacent sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

11. $\mathbf{u} = (1,1,3)$, $\mathbf{v} = (2,1,4)$, $\mathbf{w} = (5,1,-2)$

12. $\mathbf{u} = (1,3,2)$, $\mathbf{v} = (7,2,-10)$, $\mathbf{w} = (1,0,1)$

For Exercises 13-14, calculate $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ and $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$.

13. $\mathbf{u} = (1,1,1)$, $\mathbf{v} = (3,0,2)$, $\mathbf{w} = (2,2,2)$

14. $\mathbf{u} = (1,0,2)$, $\mathbf{v} = (-1,0,3)$, $\mathbf{w} = (2,0,-2)$

15. Calculate $(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{z})$ for $\mathbf{u} = (1,1,1)$, $\mathbf{v} = (3,0,2)$, $\mathbf{w} = (2,2,2)$, $\mathbf{z} = (2,1,4)$.

B

16. If $\mathbf{v}$ and $\mathbf{w}$ are unit vectors in $\mathbb{R}^3$, under what condition(s) would $\mathbf{v} \times \mathbf{w}$ also be a unit vector in $\mathbb{R}^3$? Justify your answer.

17. Show that if $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ for all $\mathbf{w}$ in $\mathbb{R}^3$, then $\mathbf{v} = \mathbf{0}$.

18. Prove Theorem 1.14(b).

19. Prove Theorem 1.14(c).

20. Prove Theorem 1.14(d).

21. Prove Theorem 1.14(e).

22. Prove Theorem 1.14(f).

23. Prove Theorem 1.16.

24. Prove Theorem 1.17. (*Hint: Expand both sides of the equation.*)

25. Prove the following for all vectors $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$:

(a) $\|\mathbf{v} \times \mathbf{w}\|^2 + |\mathbf{v} \cdot \mathbf{w}|^2 = \|\mathbf{v}\|^2 \, \|\mathbf{w}\|^2$

(b) If $\mathbf{v} \cdot \mathbf{w} = 0$ and $\mathbf{v} \times \mathbf{w} = \mathbf{0}$, then $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$.